MEMORANDUM
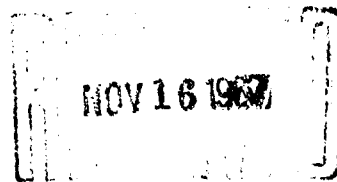RM-5433-PR
OCTOBER 1967

# AN APPRAISAL OF
# SOME SHORTEST PATH ALGORITHMS

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UNITED STATES AIR FORCE PROJECT RAND

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SANTA MONICA • CALIFORNIA

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## PREFACE

This Memorandum treats five different problems involving the determination of the shortest path through a discrete network. Previous important results are reviewed, and misleading procedures are identified and (in some cases) modified. Conclusions are drawn and recommendations are made concerning efficient algorithms.

This work forms a part of RAND's continuing interest in problems and techniques in optimization theory.

The author, a member of the Industrial Engineering faculty at the University of California, Berkeley, is a consultant to the Computer Sciences Department of The RAND Corporation.

## SUMMARY

This Memorandum treats five discrete shortest-path problems: 1) determining the shortest path between two specified nodes of a network; 2) determining the shortest paths between all pairs of nodes of a network; 3) determining the second, third, etc. shortest path; 4) determining of the fastest path through a network with travel times depending on the departure time; and 5) finding the shortest path between specified endpoints that passes through specified intermediate nodes. Existing good algorithms are identified while some others are modified to yield efficient procedures. Also certain misrepresentations and errors in the literature are demonstrated.

## CONTENTS

## I. INTRODUCTION

In the never-ending search for good algorithms for various discrete shortest-path problems, some authors have apparently overlooked or failed to appreciate previous results. Consequently, certain recently reported procedures are inferior to older ones. Also, occasionally, inefficient algorithms are adapted to new, generalized, problems where more appropriate modifiable methods already exist. Finally, the literature contains some erroneous procedures. In this Memorandum, we shall briefly consider various versions of discrete path problems in the light of known results and some original ideas, and summarize our conclusions and recommendations.

The reader should realize that our observations are by no means definitive or final. Despite this fact, it is hoped that our somewhat skeptical survey of current literature will put the interested reader on guard and perhaps save him, or his digital computer, considerable time and trouble. Since our objective is more to alert than to resolve conclusively, this Memorandum is informal and, at times, cryptic. We hope that even our most laconic remarks will become insightful for any reader deeply involved with the particular procedure.

## II.  THE SHORTEST PATH BETWEEN A SPECIFIED PAIR OF NODES

Given a set of N nodes, numbered arbitrarily from
1 to N, and the NxN matrix D, not necessarily symmetric,
whose element $d_{ij}$ represents the length of the directed
arc connecting node i to node j, find the path connecting
node 1 and node N of shortest length.  We assume initially
that $d_{ii} = 0$ and $d_{ij} \geq 0$.  If there is no arc directed
from node i to node j, then $d_{ij} = \infty$, or, for purposes of
digital computation, $d_{ij}$ is taken large.

It appears that the computationally most efficient
procedure was described first by Dijkstra [1] in 1959,[*]
and in 1960 by Whiting and Hillier.[†]  The algorithm assigns
tentative labels, which are upper bounds on the shortest
distance from node 1, to all nodes; after repeating the
fundamental iterative step described below exactly one
time for each node, the tentative node labels are all
permanent and represent actual shortest distances.  Init-
ially, node 1 is labelled with the permanent value 0, and

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[*]I am indebted to J. D. Murchland for pointing out
this reference to me.  I would appreciate being told of
any even earlier reference for this algorithm or any other
one appearing in this Memorandum.

[†]See Ref. 2, last paragraph beginning on p. 39.

all other nodes tentatively are labelled $\infty$. Then, one
by one, each node label except that at node 1 is compared
with the sum of the label of node 1 (i.e., 0) and the
direct distance from node 1 to the node in question. The
smaller of the two numbers is the new tentative label.
Next, the smallest of the N-1 tentative labels is de-
termined and declared permanent. Suppose that node k is
the one permanently labelled. Then, one at a time, each
of the remaining N-2 tentative node labels is compared to
the sum of the label just assigned permanently to node k
and the direct distance from node k to the node under
consideration. The smaller of the two numbers becomes the
tentative label. The minimum of the N-2 tentative labels
is determined, declared permanent, and made the basis of
another modification of the remaining tentative labels
of the type described above. When, after at most N-1
executions of the fundamental iterative step, node N is
permanently labelled, the procedure terminates. (If the
shortest paths from node 1 to all other nodes are desired,
the fundamental iterative step must be executed exactly
N-1 times.) The optimal paths can easily be reconstructed
if an optimal policy table (in this case, a table indicating
the node from which each permanently labelled node was
labelled) is recorded. Alternatively, no policy table

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need be constructed, since it can always be determined
from the final node labels by determining which nodes
have labels that differ by exactly the length of the con-
necting arc.

This algorithm requires $\frac{N(N-1)}{2}$ additions and $N(N-1)$
comparisons to determine all node labels. ($\frac{N(N-1)}{2}$ addi-
tions and comparisons to compute tentative node labels
during all steps and another $\frac{N(N-1)}{2}$ comparisons to pick
out the minimum one during all steps.) All steps are
naturally and easily programmed except that of recognizing
and acting on the basis of which nodes are permanently
labelled and which are tentatively labelled. Some com-
putational experimentation[*] indicates that an efficient
way to treat this consideration is to attach to each node
an index number which is changed from 0 to 1 (say) when
a node label becomes permanent. When branching out from
a just-permanently-labelled node, the index of the destina-
tion node is consulted as each outgoing arc is considered.
If the index is zero, the temporary label of the destination
is reduced, if appropriate. At the same time, memory cells
containing the smallest temporary label encountered thus

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[*] R. L. Mobley very ably programmed this and all other
computational experiments mentioned herein.

flow of one item.  In Routine 1 on p. 422 of Ref. 5, if

the times $t_{ij}$ are distances and the capacities $c_{ij}$ are

all arbitrarily set equal to 1, the node number $\pi_i$,

when i is labelled, is the shortest distance from $P_o$ to

i.  (The actual label of node i is [the node previous

to i in the shortest path to i, 1] and is irrelevant.)

By increasing the $\pi$'s associated with unlabelled nodes

by an amount chosen so that one more node can be labelled,

rather than increasing them at each iteration by one, the

"Minty algorithm" is obtained.  The method requires

approximately $N^3/6$ additions and comparisons for solving

the shortest-path problem and hence is not recommended.

Some authors have proposed simultaneously fanning out

from both endpoints as a means of reducing computation.

This advice is sometimes accompanied by the false asser-

tion that when, for the first time, some node is per-

manently labelled in both fans, the optimal path is

determined and goes through that node [6,7].  In the

problem depicted below, if the Dijkstra scheme is first

used to permanently label the node nearest A reachable

from A, then the node nearest B from which B can be

reached, then the second closest to A, etc., the node C

is permanently-labelled both "out from A" and "into B"

after two applications of the procedure at each end, yet ACB is <u>not</u> the shortest path.
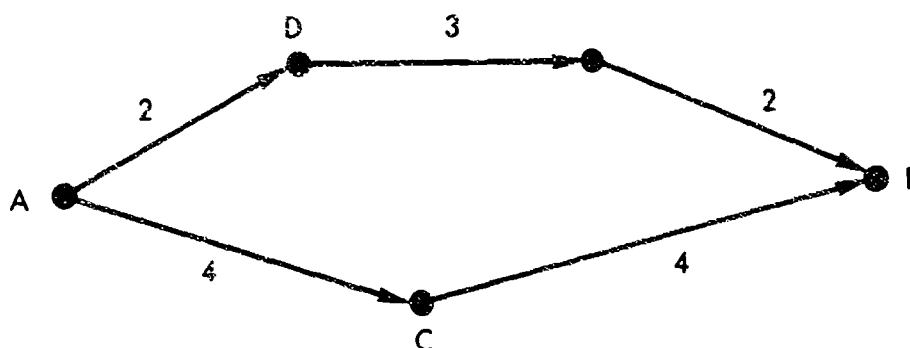


When the correct stopping procedure given cryptically by Nicholson [8] and explained clearly by Murchland [9] is used, the least upper bound on the computation required by a two-ended procedure exceeds that for the one-ended Dijkstra algorithm. This is because, as more nodes become permanently labelled in the Dijkstra procedure, a decreasing number of additions and comparisons are needed to modify all tentative labels. As a result, determining the first $\frac{N}{2}$ permanent labels requires roughly $\frac{3}{5}$ths[*] the

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[*]This number is empirical. Theory based on addition and comparison count predicts about 2/3rds, but machine-dependent and programming-language-dependent details such as how and when various address modifications are computed can cause significant variations in actual computation times. Clearly the first half of a problem requires more than half the work of complete solution, which is the important point.

work of the complete solution. If the Nicholson stopping condition is satisfied considerably before $\frac{N}{2}$ nodes have been permanently labelled from each terminus, a savings may accrue; but, in a case where close to all N nodes must be permanently labelled from either one end or the other, the two-ended procedure will prove inefficient.

All the above methods require that all elements of D be non-negative. A related problem assumes that some $d_{ij}$ are negative, but that the sum of the $d_{ij}$ around any loop is positive. (The problem has no solution if negative loops exist and are allowed to be included in paths. The problem is very difficult and no satisfactory algorithms are known if negative loops exist and the restriction is made that no node can be visited more than once.) While better methods for this problem probably are forthcoming, the author recommends the following iterative procedure, variations of which have been proposed, originally for the problem with $d_{ij} \geq 0$, by Ford [10], Moore [11], Bellman [12], and undoubtedly others. This procedure repeatedly updates all node labels (which, for the initial condition given in Eq. (1), represent, at iteration k, the lengths of the shortest paths connecting node 1 and the labelled nodes containing k+1 or less arcs)

and none are considered final until all are. If $f_i^{(k)}$ represents the label of node i at iteration k, the fundamental recursion is

$$f_i^{(k+1)} = \min_j [d_{ji} + f_j^{(k)}] \tag{1}$$

$$f_i^{(0)} = d_{1i} .$$

For the case $d_{ij} \geq 0$, convergence occurs whenever $f_i^{(k)} = f_i^{(k+1)}$ for all i, or after N-2 iterations if the former situation does not occur sooner (since no shortest path contains more than N-1 arcs). If (N-2) iterations are required, $(N-2)(N-1)^2$ additions and comparisons take place. The method is inefficient for the positive-distance problem if 2 or more iterations of equation (1) are needed; and, unless all N-2 iterations are required, as many iterations will be needed as the number of arcs in the shortest path from node 1 to node j, where node j is the node whose shortest path has the greatest number of arcs.[*]

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[*]In a personal communication, C. Witzgall points out that for networks with far less than the allowable N(N-1) arcs, the Ford iterative procedure [10] may be superior to Dijkstra's. This is because the Dijkstra algorithm, even in this case, must check indices distinguishing permanent from temporary labels for N-1 nodes at each of N-1 iterations and hence requires at least $(N-1)^2$ comparisons

(Convergence may be enhanced, and cannot be slowed, by using $f_j^{(k+1)}$ on the right in Eq. (1) as soon as it has been computed. However, given any such immediate updating scheme, examples can be constructed where no improvement in rate of convergence results.) When the above procedure is applied to the problem with some negative $d_{ij}$, either convergence will occur for $k \le N-1$, indicating no negative cycles exist and the solution is optimal, or a change in some $f_i$ will occur on the $(N-1)^{st}$ iteration, indicating the existence of a negative loop.

The above straightforward and easily programmed procedure appears slightly preferable to that given by Dantzig [13]. His scheme is unesthetic to prove, unstraightforward to program, and involves $kN^3$ ($0 < k < 1$) additions and comparisons, even when the labelling of the set T (see his Theorem 4) is carried out in the efficient fashion of Dijkstra, described above in a different context. Dantzig's own bound, based on the admittedly misleading assumption of effortless sorting, is $N^3/2$ (his Eq. (17)).

Interestingly, the two procedures recommended below for solving the shortest route problem for all pairs of

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no matter how few arcs are present; while one updating of all node labels by Ford's procedure involves only as many additions and comparisons as there are arcs.

nodes easily detect negative loops, if such exist for
the problem with some $d_{ij} < 0$, and are in some cases more
efficient for detecting negative loops than the scheme
described above. However, they are never more efficient
for establishing the optimal path between a particular
pair of nodes if there are no negative loops.

## III.  THE SHORTEST PATHS BETWEEN ALL PAIRS OF NODES
## OF A NETWORK

Two somewhat different, but equally elegant and
efficient, algorithms are recommended.  One was published
without comment as an obscure 9-line ALGOL algorithm in
1962 by Floyd [14] based on a theorem by Warshall [15]
and was rediscovered and appropriately extolled in 1965
by Murchland [16].  The other was produced in 1966 by
Dantzig [17].  Since both require exactly the same number
of calculations--N(N-2)(N-2) additions and comparisons for
the case $d_{ij} \geq 0$--are easily proved and programmed, and
culminate a steady progression of successive improvements
([18], [12], [19], [20]; actually [14] precedes the in-
ferior algorithm of [20]), there is good reason to believe
that they are definitive.  While it is recommended that
the reader consult the primary sources cited above, we
shall give very brief descriptions.

The Floyd procedure builds optimal paths by inserting
nodes, when appropriate, into more direct paths.  Starting
with the NxN matrix D of direct distances, N matrices are
constructed sequentially.  The $k^{th}$ such matrix can be
interpreted as giving the lengths of the shortest allowable
paths between all node pairs (i,j) where only paths with

intermediate nodes belonging to the set of nodes numbered from 1 through k are allowed. The $(k+1)^{st}$ matrix is constructed from the $k^{th}$ by using the formula

$$d_{ij}^{(k+1)} = \min \left[ d_{ij}^{(k)}, \ d_{i,k+1}^{(k)} + d_{k+1,j}^{(k)} \right] . \qquad (2)$$

$$d_{ij}^{(0)} = d_{ij} .$$

Here, k, which is initially zero, is incremented by 1 after i and j have ranged over the values $1,\ldots,N$ and k = N-1 at termination. To see why this procedure is valid, suppose the shortest path from node 8 to node 5 is 8 - 3 - 7 - 1 - 9 - 5. Iteration 1 will replace $d_{79}$ by $d_{71} + d_{19}$, iteration 3 will replace the current value of $d_{87}$ (which may or may not be the original value) by $d_{83} + d_{37}$ (the optimal value) iteration 7 will replace the current $d_{89}$ by $d_{87} + d_{79}$ (where these numbers are the optimal values as computed above), and iteration 9 will obtain for $d_{85}$ the sum of $d_{89}$ and $d_{95}$ when $d_{89}$ is as computed at iteration 7. Hence, the correct shortest distance is obtained. (The above indicates the proof but of course is not one.) A minor modification involving consideration of diagonal elements yields correct results

when some $d_{ij}$ are negative but no negative loops exist, as well as an indication of such if negative loops do exist. An additional advantage of this procedure is that N-1 additions and comparisons are easily circumvented whenever an element $d_{i,k+1}^{(k)}$ equals infinity in (2) (i.e., no path exists connecting nodes i and k+1 with only nodes 1 through k as intermediate nodes).[*] As in the case of the particular initial-terminal pair problem, an optimal policy table (matrix) associating with the initial-terminal node pair (i,j) the next node along the best path from i to j can be developed during the computation, or can be deduced from the final shortest-distance matrix.

Dantzig's scheme generates successive matrices of increasing size. At the $k^{th}$ iteration a k×k matrix is produced whose elements are the lengths of the shortest paths connecting nodes i and j, i=1,...,k, j=1,...,k, which contain only nodes numbered between 1 and k as intermediate nodes. Given the k×k matrix $D^{(k)}$ with elements $d_{ij}^{(k)}$ as defined above, $D^{(k+1)}$ is computed as follows:

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[*]The additional computation introduced in order to test for the presence of infinities adds about 5 percent to the computing time. For a sample problem with 10 nodes and 34 arcs and, hence, 56 infinities initially, the test yielded a 5 percent net improvement over no test. The amount of net improvement or degradation depends on the actual network configuration as well as the number of non-existent arcs.

1. Compute $d_{i,k+1}^{(k+1)}$ for $i=1,\ldots,k$ by

$$d_{i,k+1}^{(k+1)} = \min_{1\leq j\leq k}\left[d_{ij}^{(k)} + d_{j,k+1}\right] \text{, and similarly compute } d_{k+1,i}\text{.}$$

2. Compute $d_{ij}^{(k+1)}$ for $i=1,\ldots,k$, $j=1,\ldots,k$ by

$$d_{ij}^{(k+1)} = \min\left[d_{ij}^{(k)}, \, d_{i,k+1}^{(k+1)} + d_{k+1,j}^{(k+1)}\right]\text{.}$$

That the $d_{ij}^{(k)}$ yielded by the above steps are as previously defined is obvious after a little thought. (A proof is given in Ref. 17.) If all distances are positive, $d_{ii}^{(k)} = 0$ for all i and k. If not, $d_{ii}$ can be easily computed and if any $d_{ii}^{(k)}$ is negative, a negative loop exists. It does not appear that the above algorithm can exploit non-existent arcs in a manner similar to Floyd's.

If the above algorithms, requiring $N(N-1)(N-2)$ additions and comparisons, are indeed as efficient as possible, then the most efficient possible particular-pair algorithm must require at least $(N-1)(N-2)$ such calculations (assuming, as was the case in Sec. II, that such procedures must--at least in the worst case--generate best paths from the initial node to all other nodes). The best algorithm of

Sec. II, that which required no elaborate data-preparation
or list keeping, involves about $3N^2/2$ additions and com-
parisons. This gives, as a reasonable upper-bound estimate
of potential future improvements for the particular-pair-
of endpoints problem, a reduction in computation time of
33 percent. Computational experiments have confirmed that
computing optimal paths between all pairs of nodes by means
of N applications of the Dijkstra method requires one and
one-half-times the time consumed by either of the above
algorithms specifically solving the all-pairs problem.
Viewed from the perspective of a combinatorialist, the well
is rather dry. (Such is not the case for the $N^3$ procedure
recommended above for the problem with negative distances.)

## IV. DETERMINATION OF THE SECOND SHORTEST PATH

Sometimes it is desirable to know the value of the second (and third, etc.) shortest path through a network. For example, suppose that some complex quantitative (or even qualitative) feature characterizes paths, and the shortest path possessing this additional attribute is sought. By ignoring the special aspect in question and ordering paths from shortest to longest, the best path with the additional feature can sometimes be determined fairly efficiently.

In what follows, we shall initially restrict our attention to the problem of determining the second-best path between a specified initial node 1, and a specified destination, N. Then, conclusions will be drawn for more general problems. Two paths which do not visit precisely the same nodes in the same order are considered different. We exclude consideration of ties in our discussion by assuming, for simplicity, that all paths have different values. A path with a loop is considered an admissible path, and indeed such a path can be second best, even for problems with all $d_{ij} > 0$. Even node N may be visited twice along the second-best path.

The earliest good algorithm known to this author was proposed by Hoffman and Pavley [21]. A _deviation_ from the shortest path was defined to be a path that coincides with the shortest path from its origin up to some node j on the path (j may be the origin or the terminal node), then deviates directly to some node k not the next node of the shortest path, and finally precedes from k to the fixed terminal node via the shortest path from k. The second shortest path between specified initial and terminal nodes is shown in Ref. 21 to be a deviation from the shortest path. To solve the problem posed above, first the shortest paths from all initial nodes to the specified destination are determined by any efficient algorithm. Then, all deviations from the shortest path between the specified origin and terminus are determined, evaluated, and compared, and the best noted. If the average node has M outgoing links, and the average shortest path contains K arcs, an average problem is solved in approximately MK additions and comparisons beyond those required for solution of the shortest path problem.

Suppose second-shortest paths from all nodes to the specified terminal node N are sought. Then the following modification of the Hoffman-Pavley method [21] seems appropriate. After solving the shortest-path problem,

determine $v_N$, the length of the second-shortest path from N to N (it may be $\infty$), by considering all deviations at N plus best paths to N. Then, for each node k whose shortest path to N contains only one arc, compare (a) the length of the shortest path deviating at k and (b) $d_{kN} + v_N$. The minimum of these two quantities is $v_k$, the length of the second-shortest path from that node. Then, consider all nodes j which are two arcs from N via the shortest path. For each, compare (a) the length of the shortest path deviating at j and (b) the length of the arc out of j which is the first arc of the shortest path from j to N plus v evaluated at the terminal node of that arc. The minimum value is $v_j$. Repeat this iterative process until all nodes are labelled. Note that the iteration is performed on an index representing the number of arcs in the shortest path from each node. This procedure requires about MN additions and comparisons.

Reference 22, published subsequently to method 1 above, gives a seemingly different procedure. Define $u_i$ as the length of the shortest path from node i to a specified terminal node N and $v_i$ as the length of the second shortest path. Define $\min_k (x_1, \ldots, x_n)$ as the $k^{th}$ smallest value of the quantities $x_i$. Then, according to Ref. 22, $v_i$ is characterized by the equation

$$v_i = \min \begin{bmatrix} \min_2 & (d_{ij} + u_j) \\ j \neq i \\ \\ \min_1 & (d_{ij} + v_j) \\ j \neq i \end{bmatrix} , \qquad i=1,\ldots,N-1$$

$$(3)$$

$$v_N = \min_i \begin{bmatrix} d_{Ni} + u_i \end{bmatrix} .$$

(The $u_i$ are computed first, as in method 1, by any efficient procedure.)

The term $\min_2(d_{ij} + u_j)$ determines the value of the best path originating at node i and deviating from the shortest path at that node i. The term $\min_1(d_{ij} + v_j)$ determines the best path consisting of any first arc plus the second best continuation. What was not noted by the originators of the method or by Pollack [23] is that if the minimizing node in the $\min_1$ operation is not k (the next node of the already known shortest path from i) but some other node p, then $d_{ip} + u_p$ (an admissible solution to the $\min_2$ expression) is less than $d_{ip} + v_p$ (since $u_p < v_p$). Since the term $\min_1(d_{ij} + v_j)$ can only yield the overall minimum in Eq. (3) if j = k, where k is the node after i on the shortest path from i, the $\min_1$ term in Eq. (3) can be replaced by merely $d_{ik} + v_k$. After

this reduction by a factor of 2 in required computation,
we can calculate the method's approximate computational
requirements. Bellman and Kalaba [22] recommend solution
of Eq. (3) by an iterative procedure, where $v_i$ is super-
scripted on the left by $(k + 1)$ and $v_j$ on the right by $(k)$.
Defining M and N as above and L as the average number of
iterations until convergence of the iterative solution of
Eq. (3), the method requires NML additions and comparisons
on the average. L is less than N-1 and may be as large as
the number of arcs in the shortest path containing the most
arcs. However, after replacing the $min_1$ term in Eq. (3)
by $d_{ik} + v_k$ as discussed above, solution can be obtained
by a one-pass scheme first labelling nodes one-arc-by-
shortest-path from N, then two-arcs-by-shortest-path etc.
This reduces the Bellman-Kalaba [22] procedure to precisely
the modified Hoffman-Pavley [21] algorithm recommended above.

In summary, if only the second shortest path connect-
ing a _particular_ pair of nodes is desired, method 1 above
is clearly the best since it requires MK calculations
compared to MN for method 2, which ―    ,lve the all-
initial-node problem in order to solve the particular-
initial-node case. If problems involving all nodes as
initial nodes and a fixed terminal node are to be solved,
the methods as modified in this Memorandum are equivalent.

These conclusions strongly contradict those of Pollack's paper [23].

For determination of the third shortest paths from all initial nodes to N we recommend the following generalization of the above procedure. If $w_i$ represents the length of the third shortest path from i, then

$$
w_i = \min \begin{bmatrix} d_{ik} + w_k \\ \min_2 [d_{ij} + u_j] \\ j \neq i \end{bmatrix} \tag{4}
$$

if a single node k follows i along both the first and second shortest paths. If k is the node following i on the shortest path and m is the node following i on the second shortest, then

$$
w_i = \min \begin{bmatrix} d_{ik} + v_k \\ d_{im} + v_m \\ \min_3 [d_{ij} + u_j] \\ j \neq i \end{bmatrix} . \tag{5}
$$

The function $w_i$ can be computed node-by-node by first computing w at nodes that are one arc from N via the shortest

path, then two arcs, etc. This one-pass procedure generalizes to the $n^{th}$ best path problem, because the same function appears on the left and right in the appropriate equation of the type of Eq. (4) above only if the $p^{th}$ best path from i for p=1,...,n-1 all go to the same second node.

## V   TIME-DEPENDENT LENGTHS OF ARCS

At least one paper [24] has studied the problem of
finding the fastest path between cities where the time
of travel between city i and city j depends on the time
of departure from city i.  When t is the time of departure
from city i for city j, let $d_{ij}(t)$ denote the travel time.
(If travel schedules are such that a delay before departure
decreases the time of arrival, $d_{ij}(t)$ represents the
elapsed time between time t and the earliest possible
time of arrival.)  This model has applications in the areas
of transportation planning and communication routing.[*]
Cooke and Halsey [24] define $f_i(t)$ as the minimum time of
travel to N, starting at city i at time t, and establish
the formula

$$f_i(t) = \min_{j \neq i} \left[ d_{ij}(t) + f_j(t + d_{ij}(t)) \right] \qquad (6)$$

$$f_N(t) = 0 .$$

---

[*]In more realistic communication routing problems,
links are sometimes unavailable due to failure or over-
loading.  Adaptive routing techniques that adjust to net-
work changes are studied in Refs. 25 and 26.

Assuming all the $d_{ij}(t)$ are defined at, and take on, positive integer values, an iterative procedure is given for finding the quickest paths from all cities to city N, starting at city i at time 0. Defining T to be the maximum taken over all i of $d_{iN}(0)$ (a smaller T can be determined, at some inconvenience, if this number is infinite), and assuming all cities are connected at all times (perhaps by links taking infinite time), the procedure requires at most $N^2 T^2$ additions and comparisons.

We wish to point out here that this problem can be solved by the method by Dijkstra [1] discussed in Sec. II above just as efficiently as can the problem where the times (or distances) are not time-dependent. Also, the restriction to integer-valued times can be dropped and any real-valued times can be treated. Define the tentative node (city) label $f_i$ to be an upper bound on the earliest time of arrival at node i, and permanent labels to be earliest possible (optimal) times-of-arrival. First, permanently label node $i_0$ (the initial node) zero and all other nodes infinity. Next, tentatively label all nodes j with the minimum of the current node label $f_j$ and the sum of $f_{i_0}$ and $d_{i_0 j}(f_{i_0})$. Then, find the minimum, non-permanent node label, say $f_k$, and declare it permanent.

($f_k$ is the earliest possible time of arrival at node k,
leaving node $i_o$ at time 0.) Node k is then used to
possibly reduce the labels at all tentatively labelled
cities, by comparing $f_k + d_{kj}(f_k)$ to the current label,
and the minimum new temporary label is made permanent,
etc. After at most $N^2$ comparisons and $\frac{N^2}{2}$ additions, city
N is labelled, and the quickest paths, leaving $i_o$ at time
0, to all nodes, including N, are determined. As is the
case for the closely related Dijkstra procedure [1],
about $3N^2/2$ additions and comparisons are required when a
method of distinguishing temporarily from permanently-
labelled nodes is implemented. If quickest paths from all
cities to N are desired, the algorithm must be repeated
N-1 times; but, even then, the procedure compares quite
favorably, with respect to both required assumptions and
computation, with the Cooke-Halsey algorithm [24].

## VI.  SHORTEST PATHS VISITING SPECIFIED NODES

Given a set of N nodes and distances $d_{ij} \geq 0$, suppose
it is desired to find the shortest path between nodes 1
and N which passes through the k-1 nodes 2, 3,...,k ≤ N-1,
called "specified nodes."  A simple, but completely
erroneous, solution of this problem was reported by Saksena
and Kumar [27].  Noting this, we wish to give a solution
method.

The fallacy in Ref. 27 is the assertion (subject to
a proviso to follow) that the shortest path from a specified
node i to N passing through at least p of the specified nodes
enroute is composed of the shortest path from i to some
specified node j followed by the shortest path from j to N
passing through at least p-r-1 specified nodes, where r is
the number of specified nodes that turn out to lie on the
shortest unrestricted path from i to j.  Saksena and Kumar
[27] erroneously assert that this is true provided--should
specified nodes occurring on the shortest path from i to
j also lie on the continuation path from j to N and there-
fore be counted twice--that at least p distinct specified
nodes lie on the path.  Should some candidate path corres-
ponding to some j violate the duplication of nodes proviso
above, that possibility is inadmissible and going initially
from i to j is dropped from consideration.  The procedure

fails to note that, in this case, some less short continua-
tion from j passing through at least p-r-1 specified nodes
and avoiding duplication of nodes may yield a better path
than the best remaining path satisfying the conditions de-
scribed above.  For example, in the network shown below,
with all nodes considered specified, suppose we seek the
best path from 1 to 5 passing through at least two inter-
mediate nodes.
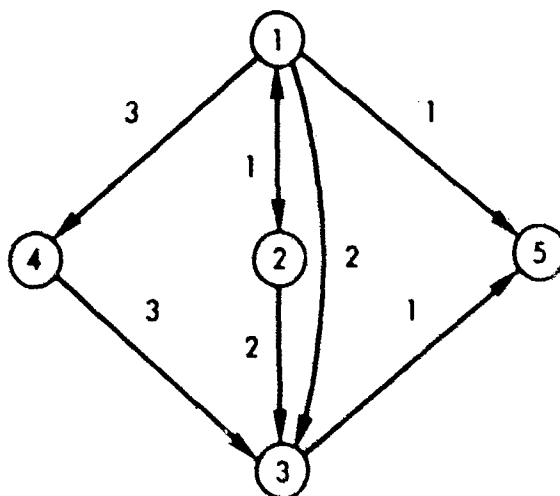


The best path from 1 to 4 has length 3 and passes through
no nodes enroute, and the best from 4 to 5 passing through
at least one node has length 4; hence, this possibility has
length 7.  The best path from 1 to 3 and best continuation
from 3 passing through one node has length infinity (no

such continuation from 3 exists). The best path from 1 to 2 has length 1 and the best continuation, passing through one node enroute, from 2 to 5 has length 2 (it returns to node 1) yielding a sum of 3. However, this possibility is eliminated by the fact that node 1 is counted twice. The answer, by the Saksena-Kumar algorithm [27] would then be 7, the best of the other alternatives. Yet, the path 1-2-3-5 has length 4 and is admissible. This is an example of a best first portion and a second-best continuation being optimal. (Or, the same path can alternatively be viewed as the second-best path from 1 to 3 followed by the best continuation.) No simple modification of the referenced method seems to handle this kind of situation.

Assuming paths with loops are admissible, the problem can be correctly solved as follows. First solve the shortest path problem for the N-node network for all pairs of initial and final nodes. Let $d'_{ij}$ represent the length of the shortest path from node i to j. Then, solve the (k + 1)-city "traveling-salesman" problem for the shortest path from 1 to N passing through nodes 2, 3,...,k where the distance from node i to j is $d'_{ij}$. Methods of solution are discussed by Gomory [28]. While there are no easy solutions for the traveling-salesman problem, it is hard to see how

the specified-city problem can possibly be any easier
than the traveling-salesman problem of dimension $k + 1$
since if $k = N-1$ it _is_ the traveling-salesman problem.

## VII.  CONCLUSION

We have given some observations and recommendations
for computationally treating discrete shortest-path problems
of varying degrees of generality.  Our intent has not been
definitive solution, but rather to clear the air by pre-
senting both some important methods and references and some
critical comments and warnings.

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